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Robust multipartite thermal entanglement

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Abstract
Recently, Mintert *et al* (MKB) (2005 *Phys. Rev. Lett.* **95** 260502) proposed generalizations of the Wootters concurrence for multipartite quantum systems, which can be evaluated efficiently for arbitrary mixed states. In this paper, we study in detail the origin of the robustness of multipartite thermal entanglement in N-qubit XY models with $N = 2, 4$ and 6 using the MKB concurrences. For $N = 2$, we establish the principle for the occurrence of robust bipartite thermal entanglement (Kamta and Starace 2002 *Phys. Rev. Lett.* **88** 107901). We show explicitly that similar robust multipartite thermal entanglement can be found in the $N = 4$ case, and how this can be explained employing the same principle. It is then deduced that the six-qubit XY model also possesses robust multipartite thermal entanglement like the $N = 2$ case. More generally, our work spells out the characteristic properties for a given realistic model to have robust multipartite thermal entanglement like that in the two-qubit XY model.

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1. Introduction

An intriguing phenomenon in a quantum system composed of more than one component is entanglement. It refers to the non-classical correlations that exist among these subsystems. Due to this nature, quantum entanglement has recently been recognized as a physical resource for performing classically impossible information processing tasks, such as quantum computation [1]. Indeed, for pure-state quantum computation, multipartite entanglement is a necessary resource for quantum computational speedups [2]. Proposals for realizing quantum computation using quantum dots (localized electron spins) as qubits [3] have revived interest in various spin models that have been extensively studied in solid state physics (see references in [4]). In an interesting paper [5], Kamta and Starace showed that the anisotropy and magnetic field may together be used to control the extent of thermal entanglement[1] in a two-qubit

[1] The state of a typical solid-state system in thermal equilibrium with some heat bath at temperature T is mixed and given by $\chi = e^{-\beta H}/Z$, where H is the Hamiltonian, $Z = \mathrm{tr}\, e^{-\beta H}$ is the partition function and $\beta = 1/kT$ with k as the Boltzmann constant. The entanglement associated with the thermal state χ is referred to as the thermal entanglement [4].

XY model and, especially, to produce entanglement for any finitely large T, by adjusting the external magnetic field beyond some finitely large critical strength. Such robustness[2] is absent in the case of a two-qubit isotropic XX model [6]. Natural questions are whether thermal entanglement of an N-qubit XY model (with arbitrary N) has similar behavior, and what is the origin of such robustness. More generally, one would like to know if similar robustness exists in any other realistic models too. These questions are not only of practical but also of fundamental importance (see [7] and references therein).

To answer the above questions, it is necessary to quantify the amount of entanglement that is associated with a given multipartite state—a subject for which we have yet to achieve a complete understanding Recently, Mintert, Kuś and Buchleitner (MKB) [8, 9] proposed generalizations of the Wootters concurrence [10][3] for multipartite quantum systems, which can be evaluated efficiently for arbitrary mixed states. Specifically, for two qubits, the MKB concurrence coincides with the Wootters concurrence. This has led to the direct measurement of the Wootters concurrence for pure states in the laboratory [11]. It is a remarkable result as the Wootters concurrence was previously thought to be not directly measurable.

In this paper we spell out, using the MKB concurrences, the characteristic properties for a given realistic model to have robust multipartite thermal entanglement like that in the two-qubit XY model. In particular, we consider the N-qubit XY models with $N = 2$, 4 and 6. For the $N = 2$ case, we establish via equation (8) the principle for the occurrence of robust bipartite thermal entanglement. Note that previously, with the Wootters concurrence, it is not clear what exactly are the key properties of the two-qubit XY model that cause it to have this desirable behavior. We show explicitly that similar robust multipartite thermal entanglement can be found in the $N = 4$ case, and how this can again be accounted for by the same principle. It is then deduced that the six-qubit XY model also possesses robust multipartite thermal entanglement like the $N = 2$ case.

Our paper is organized as follows. In section 2, we briefly review the results of MKB [8, 9] for N-qubits that are relevant to our subsequent analysis. A short introduction to the N-qubit XY model is given in section 3. Our results are presented in section 4. We conclude in section 5 with some comments.

2. The MKB concurrences

Consider an N-qubit pure state $|\psi\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_N$; the MKB concurrence is defined as the expectation value of a Hermitian operator A that acts on two copies of the state [8]:

$$\mathcal{C}[|\psi\rangle\langle\psi|] \equiv \sqrt{\langle\psi| \otimes \langle\psi|A|\psi\rangle \otimes |\psi\rangle}. \tag{1}$$

In general, with $p_{s_1 s_2 \cdots s_N} \geqslant 0$,

$$A \equiv \sum_{s_1 s_2 \cdots s_N} p_{s_1 s_2 \cdots s_N} P_{s_1}^{(1)} \otimes P_{s_2}^{(2)} \otimes \cdots \otimes P_{s_N}^{(N)}.$$

$s_m = \{+, -\}$ and the above summation is performed over the set $\{s_1 s_2 \cdots s_N\}^+$, which contains all N-long strings of $+$'s and $-$'s. The superscript $+$ indicates that the string with $N+$'s is excluded from the sum. $P_+^{(m)} \equiv \frac{1}{2}\left(\Pi_0^+ + \Pi_1^+ + \Pi_0^-\right)$ and $P_-^{(m)} \equiv \frac{1}{2}\Pi_1^-$ are respectively projectors onto the symmetric and antisymmetric subspaces of $\mathcal{H}_m \otimes \mathcal{H}_m$. That

[2] By robustness, we mean the endurance of entanglement against noise (like in [22]). Specifically, robustness refers to the endurance of multipartite entanglement against thermal noise in this work.

[3] Consider a two-qubit state ρ, the Wootters concurrence $\mathcal{C}_W[\rho] \equiv \max\{\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4, 0\}$, where λ_k ($k = 1, 2, 3, 4$) are the square roots of the eigenvalues in decreasing order of magnitude of the spin-flipped density-matrix operator $R = \rho(\sigma^y \otimes \sigma^y)\rho^*(\sigma^y \otimes \sigma^y)$, the asterisk indicates complex conjugation.

is, $\Pi_0^{\pm} \equiv (|00\rangle \pm |11\rangle)(\langle 00| \pm \langle 11|)$, $\Pi_1^{\pm} \equiv (|01\rangle \pm |10\rangle)(\langle 01| \pm \langle 10|)$, where $\{|0\rangle, |1\rangle\}$ is an orthonormal basis of $\mathcal{H}_m$. By choosing the value of all the $p_{s_1 s_2 \cdots s_N}$'s to be 4, an entanglement monotone $\mathcal{C}_N$ can be obtained. The resulting operator A_N can equivalently be written as $4\big(I - P_+^{(1)} \otimes P_+^{(2)} \otimes \cdots \otimes P_+^{(N)}\big)$. The concurrence $\mathcal{C}_N$ of an N-qubit pure state $|\psi\rangle$ can then be written as [8, 9]

$$\mathcal{C}_N[|\psi\rangle\langle\psi|] = 2^{1-N/2}\sqrt{(2^N - 2)\langle\psi|\psi\rangle - \sum_i \operatorname{Tr}\rho_i^2}. \tag{2}$$

The above summation runs over all $(2^N - 2)$ reduced density operators ρ_i of the state $|\psi\rangle$. $\mathcal{C}_N$ takes the value zero if and only if the state is fully separable. For an even number N of qubits, it is possible to define another MKB concurrence $\mathcal{C}^{(N)}$ that detects multipartite entanglement, by choosing the operator $A = A^{(N)} \equiv 2^N P_-^{(1)} \otimes P_-^{(2)} \otimes \cdots \otimes P_-^{(N)}$. We note that when $N = 2$,

$$\mathcal{C}^{(2)}[|\psi\rangle\langle\psi|] = |\langle\psi^*|\sigma^y \otimes \sigma^y|\psi\rangle| = \mathcal{C}_W[|\psi\rangle\langle\psi|]. \tag{3}$$

That is, the MKB concurrence $\mathcal{C}^{(2)}$ coincides with the Wootters concurrence $\mathcal{C}_W$ [10]. And for $N = 4$, we have

$$\mathcal{C}^{(4)}[|\psi\rangle\langle\psi|] = |\langle\psi^*|\sigma^y \otimes \sigma^y \otimes \sigma^y \otimes \sigma^y|\psi\rangle|, \tag{4}$$

which, like $\mathcal{C}^{(2)}$, is an entanglement monotone [9]. Since the expectation value of $A^{(N)}$ is zero when N is odd, there is no equivalent definition for odd number of qubits.

The MKB concurrence for a mixed state ρ of N qubits can be obtained via the convex roof construction [12]:

$$\mathcal{C}[\rho] \equiv \inf\left\{\sum_i p_i \mathcal{C}[|\psi_i\rangle\langle\psi_i|], \rho = \sum_i p_i|\psi_i\rangle\langle\psi_i|\right\}, \tag{5}$$

where the infimum is taken over all possible pure state decompositions of ρ. To evaluate $\mathcal{C}[\rho]$, consider the spectral decompositions $\rho = \sum_j |\tilde{\phi}_j\rangle\langle\tilde{\phi}_j|$ and $A = \sum_\alpha |\tilde{\chi}_\alpha\rangle\langle\tilde{\chi}_\alpha|$, where the eigenstates $|\tilde{\phi}_j\rangle$ and $|\tilde{\chi}_\alpha\rangle$ are subnormalized such that their norms squared are the eigenvalues corresponding to the states. If r is the rank of the operator A, it is possible to define r complex symmetric matrices T^α with elements $T^\alpha_{jk} \equiv \langle\tilde{\phi}_j| \otimes \langle\tilde{\phi}_k|\tilde{\chi}^\alpha\rangle$. And, equation (5) becomes

$$\mathcal{C}[\rho] = \inf_V \sum_i \sqrt{\sum_\alpha |[VT^\alpha V^T]_{ii}|^2}, \tag{6}$$

where the infimum is now taken over the set of left unitary matrices V. It can be shown that the following inequality holds [8]:

$$\mathcal{C}[\rho] \geqslant \inf_V \sum_i |[V\tau V^T]_{ii}|. \tag{7}$$

The matrix τ is defined to be $\sum_\alpha z_\alpha T^\alpha$ in terms of arbitrary complex numbers z_α satisfying only the condition that $\sum_\alpha |z_\alpha|^2 = 1$. An algebraic solution of the inequality equation (7) is given in [8] to be $\max\left\{0, \lambda_1 - \sum_{j>1}\lambda_j\right\}$ where λ_j's are singular values of τ written in decreasing order. For $\mathcal{C}^{(N)}$, where $A = A^{(N)}$ is of rank 1, $T^1 = \tau$ and the lower bound in equation (7) turns out to be the exact value of $\mathcal{C}^{(N)}$.

In general, an optimization over z_α is also necessary to obtain the optimal lower bound for $\mathcal{C}[\rho]$. However, it is possible to obtain a good approximation to $\mathcal{C}[\rho]$ by approximating τ with a matrix whose elements [8] are

$$\tau_{ij} \approx \frac{\langle\tilde{\phi}_1| \otimes \langle\tilde{\phi}_1|A|\tilde{\phi}_i\rangle \otimes |\tilde{\phi}_j\rangle}{\sqrt{\langle\tilde{\phi}_1| \otimes \langle\tilde{\phi}_1|A|\tilde{\phi}_1\rangle \otimes |\tilde{\phi}_1\rangle}}, \tag{8}$$

where $|\tilde{\phi}_1\rangle$ is the eigenstate of ρ with the largest eigenvalue. This is the mathematical property of MKB concurrences that will play a crucial role in our understanding of the origin of the characteristic robustness of multipartite thermal entanglement in the N-qubit XY model. In particular, equation (8) together with equation (3) allows one to understand, for the first time, the characteristic properties of the two-qubit XY model that give rise to robust bipartite thermal concurrence [5].

3. The N-qubit XY model

The Hamiltonian for the anisotropic N-qubit XY model in an external magnetic field $B \equiv \eta J$ (η is a real number) along the z-axis is [5]

$$H_N = \frac{J}{2}\sum_{m=1}^{N}\left[(1+\gamma)\sigma_m^x\sigma_{m+1}^x + (1-\gamma)\sigma_m^y\sigma_{m+1}^y + \eta\sigma_m^z\right]. \tag{9}$$

$\sigma_m^\alpha(\alpha = x, y, z)$ are the Pauli matrices at site m, and the periodic boundary condition $\sigma_{N+1}^\alpha = \sigma_1^\alpha$ applies. The parameter $-1 \leqslant \gamma \leqslant 1$ measures the anisotropy of the system. It equals 0 for the isotropic XX model and ± 1 for the Ising model. $(1+\gamma)J$ and $(1-\gamma)J$ are real coupling constants for the spin interaction. The model is said to be antiferromagnetic for $J > 0$ and ferromagnetic for $J < 0$. Without loss of generality, we consider here the case when $\eta \geqslant 0$, $0 \leqslant \gamma \leqslant 1$ and $J > 0$.

For the above system in thermal equilibrium at temperature T, its state is described by the density operator

$$\chi_N = \sum_{i=1}^{2^N} w_i|\Phi^i\rangle\langle\Phi^i|. \tag{10}$$

The statistical weights $w_i \equiv \exp(-\beta E_i)/Z_N$, where the partition function $Z_N = \sum_{i=1}^{2^N}\exp(-\beta E_i)$. Boltzmann's constant $k \equiv 1$ from here on and $\beta = 1/T$. $|\Phi^i\rangle$'s are the eigenvectors of H_N with corresponding eigenvalues E_i's. We note that, as $T \to \infty$, $w_i \to 2^{-N}$ for all i, and χ_N becomes the maximally mixed state with neither classical nor quantum correlations. Therefore, there exist critical temperatures T_c beyond which the entanglement of the system becomes zero. The existence of T_c's is guaranteed by the fact that a state becomes separable when it is sufficiently close to the maximally mixed state [13].

4. Results

4.1. $N = 2$

The point of this subsection is to establish the principle of robust bipartite thermal entanglement in the two-qubit XY model. To this end, we present briefly the key results of [5]. The eigenvalues of H_2 are $\pm\mathcal{B}$ and $\pm J$, where $\mathcal{B} \equiv \sqrt{B^2+\gamma^2J^2} = \sqrt{\eta^2+\gamma^2}J$. In the limit of large η,

$$|\Phi^g\rangle = \frac{1}{\sqrt{(\mathcal{B}-B)^2+\gamma^2J^2}}[(\mathcal{B}-B)|00\rangle - \gamma J|11\rangle] \tag{11}$$

is the eigenstate of H_2 with the minimum eigenvalue $E_g = -\mathcal{B}$. The Wootters concurrence associated with $|\Phi^g\rangle$ is given by $\gamma/\sqrt{\eta^2+\gamma^2}$. $|\Phi^g\rangle$ is thus entangled if $\gamma \neq 0$, and for large η,

$$\mathcal{C}_W[|\Phi^g\rangle\langle\Phi^g|] \approx \gamma\eta^{-1}. \tag{12}$$

It goes asympotically to zero only when η is infinitely large.

For a fixed $\eta = \eta_0$ and non-zero temperatures, $\mathcal{C}_W[\chi_2]$ decreases to zero as the temperature T is increased beyond some critical value T_c due to mixing. However, for any finite $T > T_c$, we note that

$$\mathcal{C}_W[\chi_2] \approx \gamma\eta^{-1}, \tag{13}$$

if η is increased beyond η_0 and is large enough. Clearly, the thermal concurrence of the system in this case is very well approximated by the concurrence of $|\Phi^g\rangle$.

To understand the above result, we first recall the equivalence between the Wootters concurrence and the MKB concurrence $\mathcal{C}^{(2)}[\chi_2]$ (equation (3)). Next, we observe that for a given T, the statistical weight corresponding to $|\Phi^g\rangle$,

$$w_g \equiv \frac{\mathrm{e}^{-\beta E_g}}{Z_2} = \frac{1}{1 + \mathrm{e}^{-2\beta\mathcal{B}} + \mathrm{e}^{-\beta(\mathcal{B}-J)} + \mathrm{e}^{-\beta(\mathcal{B}+J)}}, \tag{14}$$

can be made as close to unity as possible by increasing the strength η of the external magnetic field. Lastly, we recall equation (8), which quantitatively describes how the entanglement associated with the thermal state χ_2 comes mainly from that associated with $|\Phi^g\rangle$. So, when η is large enough, only $|\Phi^g\rangle$ contributes significantly to the thermal concurrence of χ_2. The η required for this to occur depends on T, larger η for higher T. More importantly, this is always possible as long as T is finite.

In summary, the characteristic robustness of thermal entanglement in the two-qubit XY model is due to the following two facts. First, for any non-zero T and a suitably large η, $|\Phi^g\rangle$ can be made the highest weight eigenstate. Second, $|\Phi^g\rangle$ remains entangled, that is, $\mathcal{C}^{(2)}[\Phi^g\rangle\langle\Phi^g|] \neq 0$ at this η. We note that if $\gamma = 0$, then $|\Phi^g\rangle$ is a product state with no entanglement. This explains why we do not observe such robustness for the two-qubit XX model [6].

4.2. $N = 4$

In this subsection, we present our main results. We begin with the spectral decomposition of H_4. The eigenvalues of H_4 are 0, $\pm\eta J$, $\pm(2 \pm \sqrt{\eta^2 + 4\gamma^2})J$, and $\pm\omega^\pm J$ with

$$\omega^\pm \equiv \sqrt{2}\sqrt{[\eta^2 + 2(1+\gamma^2)] \pm \sqrt{[\eta^2 + 2(1+\gamma^2)]^2 - 8\eta^2}}.$$

As in the $N = 2$ case, it suffices to analyze the ground state(s) in detail. In the zero temperature limit, $T \to 0$ ($\beta \to \infty$), depending on the strength of the applied magnetic field, there are two possible lowest energy eigenstates. Corresponding to the eigenvalue $E_{g_0} = -\omega^+ J$, we have

$$\begin{aligned}|\Phi^{g_0}\rangle = N_\Omega(&\Omega_1|0000\rangle + \Omega_2|0011\rangle + \Omega_3|0101\rangle + \Omega_2|0110\rangle \\ &+ \Omega_2|1001\rangle + \Omega_3|1010\rangle + \Omega_2|1100\rangle + |1111\rangle),\end{aligned} \tag{15}$$

with $N_\Omega \equiv 1/\sqrt{1 + \Omega_1^2 + 4\Omega_2^2 + 2\Omega_3^2}$, $\Omega_1 = [(2\eta-\omega^+)((\omega^+)^2-8)-8\gamma^2(\eta-\omega^+)]/8\gamma^2\eta$, $\Omega_2 = (2\eta-\omega^+)/4\gamma$ and $\Omega_3 = -(2\eta-\omega^+)/\gamma\omega^+$. And corresponding to $E_{g_1} = -(2+\sqrt{\eta^2+4\gamma^2})J$,

$$\begin{aligned}|\Phi^{g_1}\rangle = \frac{1}{2\sqrt{1+\alpha^2}}(&-\alpha|0001\rangle + \alpha|0010\rangle - \alpha|0100\rangle + |0111\rangle \\ &+ \alpha|1000\rangle - |1011\rangle + |1101\rangle - |1110\rangle),\end{aligned} \tag{16}$$

with $\alpha \equiv (\sqrt{\eta^2 + 4\gamma^2} - \eta)/2\gamma$. For a given γ, as η is increased from zero, there are in general two instances when $E_{g_0} = E_{g_1}$. We let η_1 and η_2 denote the solutions. Their dependences on the anisotropy of the system are plotted in figure 1. As γ is increased from zero, both η_1 and

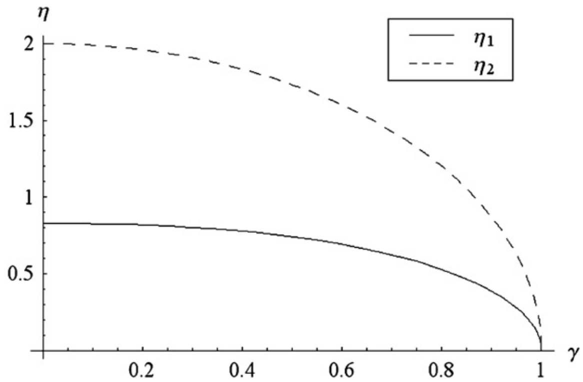

Figure 1. The two transition η's for the system at zero temperature are plotted against the anisotropy γ.

η_2 become smaller and converge to zero when $\gamma = 1$. For $0 < \gamma < 1$, the density operator of the system

$$\chi_4 = \begin{cases} |\Phi^{g_0}\rangle\langle\Phi^{g_0}| & 0 \leqslant \eta < \eta_1 \\ \frac{1}{2}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}| + |\Phi^{g_1}\rangle\langle\Phi^{g_1}|] & \eta = \eta_1 \\ |\Phi^{g_1}\rangle\langle\Phi^{g_1}| & \eta_1 < \eta < \eta_2 \\ \frac{1}{2}[|\Phi^{g_1}\rangle\langle\Phi^{g_1}| + |\Phi^{g_0}\rangle\langle\Phi^{g_0}|] & \eta = \eta_2 \\ |\Phi^{g_0}\rangle\langle\Phi^{g_0}| & \eta > \eta_2. \end{cases} \tag{17}$$

For the Ising model ($\gamma = 1$), $|\Phi^{g_0}\rangle$ and $|\Phi^{g_1}\rangle$ are degenerate if $\eta = 0$. The system state is an equal mixture of both states. But, once the external magnetic field is turned on, $|\Phi^{g_0}\rangle$ becomes the ground state of the model regardless of the strength of the field.

The MKB concurrences, defined in section 2, for $|\Phi^{g_0}\rangle$ are given by

$$\begin{aligned} \mathcal{C}^{(4)}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|] &= 2N_\Omega^2\left(\Omega_1 + 2\Omega_2^2 + \Omega_3^2\right), \\ \mathcal{C}_4[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|] &= N_\Omega^2\sqrt{7\left(2\Omega_2^2 + \Omega_3^2\right)^2 + 2\left(4\Omega_1^2 - \Omega_1 + 4\right)\left(2\Omega_2^2 + \Omega_3^2\right) + 7\Omega_1^2}. \end{aligned} \tag{18}$$

And, those for $|\Phi^{g_1}\rangle$ are given by

$$\mathcal{C}^{(4)}[|\Phi^{g_1}\rangle\langle\Phi^{g_1}|] = \frac{2\alpha}{1+\alpha^2}, \qquad \mathcal{C}_4[|\Phi^{g_1}\rangle\langle\Phi^{g_1}|] = \frac{1}{1+\alpha^2}\sqrt{\frac{3 + 8\alpha^2 + 3\alpha^4}{2}}. \tag{19}$$

We note that, for non-zero γ, both $|\Phi^{g_0}\rangle$ and $|\Phi^{g_1}\rangle$ are states with genuine four-partite entanglement[4]. When $\gamma = 0$, $|\Phi^{g_1}\rangle$ reduces to a W state [14]. In this case, $\mathcal{C}^{(4)}[|\Phi^{g_1}\rangle\langle\Phi^{g_1}|] = 0$ in agreement with the fact that a W state is not genuinely four-partite entangled [15, 16].

It follows that we can determine the MKB concurrrences for different magnetic field strengths, when the system is at temperature $T = 0$. Figures 2 and 3 respectively show plots of $\mathcal{C}^{(4)}$ and $\mathcal{C}_4$ versus magnetic field $B = \eta J$ for different values of the anisotropy parameter γ. Sharp changes in $\mathcal{C}^{(4)}$ (and in $\mathcal{C}_4$) occur at η_1 and η_2, which indicate level crossings.

For $T \neq 0$, figure 4 shows the critical temperatures T_c above which the MKB concurrence $\mathcal{C}^{(4)}$ becomes zero. Like the two-qubit case, it can be seen from figure 4 that we always have non-zero thermal entanglement by applying a sufficiently large magnetic field. That is,

[4] It turns out that $\mathcal{C}^{(4)}[|\Psi\rangle\langle\Psi|] \neq 0$ is only a sufficient condition for $|\Psi\rangle$ to have genuine multipartite entanglement. However, in this case, it can be shown that $|\Phi^{g_0}\rangle$ and $|\Phi^{g_1}\rangle$ possess genuine four- partite entanglement [15, 16].

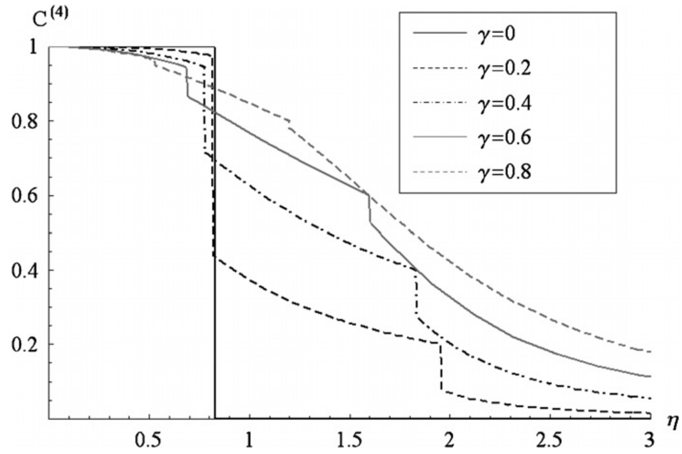

Figure 2. $\mathcal{C}^{(4)}$ versus η for different values of γ.

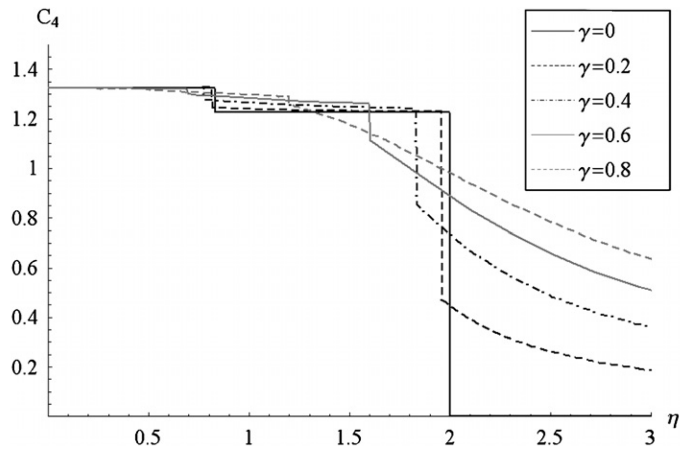

Figure 3. $\mathcal{C}_4$ versus η for different values of γ.

when $T > T_c$ for an η_0, one could ensure that the system stays entangled by increasing η. This is possible for any finite T provided $\gamma \neq 0$. Hence, multipartite thermal entanglement associated with the four-qubit XY model has characteristic robustness similar to the $N = 2$ case. However, we note that, in contrast (see figure 3 in [5]), each of our graphs in figure 4 has two 'singular turning points'. This is due to the fact that there are two transition η's instead of one in the two-qubit XY model.

In order to understand this robustness, we draw inspiration from the $N = 2$ case. First, we note that in the limit of large $\eta > \eta_2$

$$\mathcal{C}^{(4)}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|] \approx \frac{2\gamma^2}{\eta^2} + \frac{8\gamma^2 - 4\gamma^4}{\eta^4}, \tag{20}$$

$$\mathcal{C}_4[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|] \approx \frac{2\gamma}{\eta} + \frac{24\gamma - 9\gamma^3}{4\eta^3}, \tag{21}$$

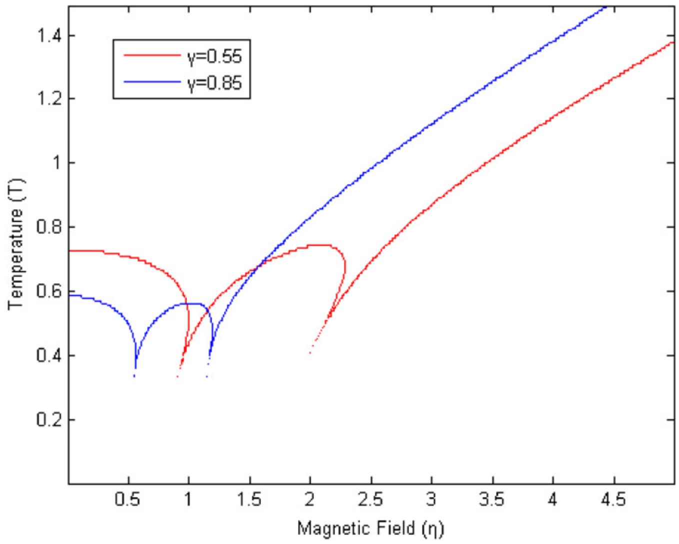

Figure 4. Temperature at which $\mathcal{C}^{(4)}$ vanishes is plotted against the magnetic field strength for two different values of the anisotropy parameter γ. The region above each curve is where $\mathcal{C}^{(4)} = 0$. (This figure is in colour only in the electronic version)

which remain non-zero even for large η provided $\gamma \neq 0$. They go to zero only in the asymptotic limit of infinite magnetic field. Next, we observe that for any finite temperature T, the statistical weight corresponding to $|\Phi^{g_0}\rangle$,

$$w_{g_0} = \frac{e^{\beta\omega^+ J}}{Z_4} = \frac{1}{\xi} \tag{22}$$

where $\xi \equiv 1 + e^{-2\beta\omega^+ J} + 4e^{-\beta\omega^+ J} + 4e^{-\beta\omega^+ J}\cosh\beta\eta J + 2e^{-\beta\omega^+ J}\cosh\beta(2 + \sqrt{\eta^2 + 4\gamma^2})J + 2e^{-\beta\omega^+ J}\cosh\beta(2 - \sqrt{\eta^2 + 4\gamma^2})J + 2e^{-\beta\omega^+ J}\cosh\beta\omega^- J$, can be made as close as possible to unity by increasing η. It can easily be shown that this large η behavior of w_{g_0} is independent of γ. Consequently, $|\Phi^{g_0}\rangle$ is the only state that significantly contributes to the thermal concurrence of χ_4. In the light of equation (8), we may conclude that

$$\mathcal{C}^{(4)}[\chi_4] \approx \mathcal{C}^{(4)}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|]. \tag{23}$$

The above conclusion can be established numerically since the MKB concurrences $\mathcal{C}^{(4)}[\chi_4]$ and $\mathcal{C}^{(4)}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|]$ are exactly computable. For a given temperature T, the two concurrences can be compared. $\mathcal{C}^{(4)}[\chi_4]$ and $\mathcal{C}^{(4)}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|]$ indeed always agree when a large enough magnetic field is applied (see table 1).

Figure 5 explicitly shows the revival of $\mathcal{C}^{(4)}[\chi_4]$ for a temperature above a high enough T_c [5]. Two effects manifest here when η is increased. Initially, increasing η results in $|\Phi^{g_0}\rangle$ becoming more dominant in the thermal mixture, leading to an increase in $\mathcal{C}^{(4)}[\chi_4]$. From figure 2, we observe that $\mathcal{C}^{(4)}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|]$ however decreases when η is increased. Clearly, beyond a certain value of η, this effect will become more important than the former, and $\mathcal{C}^{(4)}[\chi_4]$ starts to decrease.

In summary, we have analyzed the multipartite entanglement associated with the four-qubit XY model at zero and non-zero temperatures using the MKB concurrences. For

[5] This choice allows one to avoid complications due to level crossings caused by increasing η.

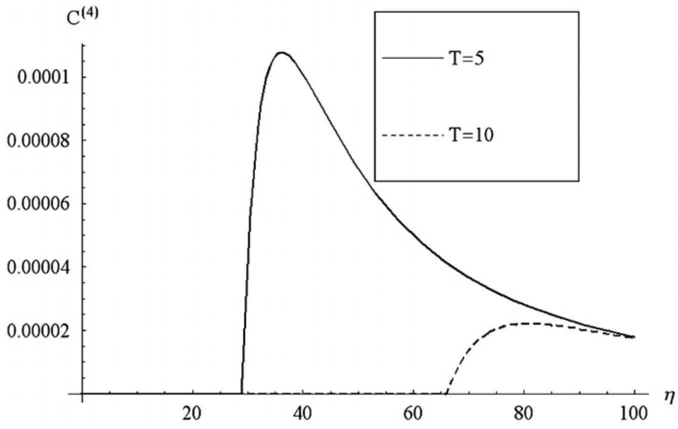

Figure 5. $\mathcal{C}^{(4)}$ versus η for a temperature $T > T_c$. The values of J and η are 1 and 0.3, respectively.

Table 1. Numerical comparison of $\mathcal{C}^{(4)}[\chi_4]$ and $\mathcal{C}^{(4)}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|]$ with $J = 1$ and $\gamma = 0.3$ at selected values of η and T.

	$\eta = 0$		$\eta = 100$		$\eta = 1000$	
T	$\mathcal{C}^{(4)}[\chi_4]$	$\mathcal{C}^{(4)}[\lvert\Phi^{g_0}\rangle\langle\Phi^{g_0}\rvert]$	$\mathcal{C}^{(4)}[\chi_4]$	$\mathcal{C}^{(4)}[\lvert\Phi^{g_0}\rangle\langle\Phi^{g_0}\rvert]$	$\mathcal{C}^{(4)}[\chi_4]$	$\mathcal{C}^{(4)}[\lvert\Phi^{g_0}\rangle\langle\Phi^{g_0}\rvert]$
1	0.000 00	1.000 00	$1.800\,69\times 10^{-5}$	$1.800\,69\times 10^{-5}$	$1.791\,77\times 10^{-7}$	$1.791\,77\times 10^{-7}$
5	0.000 00	1.000 00	$1.800\,68\times 10^{-5}$	$1.800\,69\times 10^{-5}$	$1.791\,77\times 10^{-7}$	$1.791\,77\times 10^{-7}$
10	0.000 00	1.000 00	$1.743\,16\times 10^{-5}$	$1.800\,69\times 10^{-5}$	$1.791\,77\times 10^{-7}$	$1.791\,77\times 10^{-7}$
50	0.000 00	1.000 00	0.000 00	$1.800\,69\times 10^{-5}$	$1.791\,75\times 10^{-7}$	$1.791\,77\times 10^{-7}$

$\gamma \neq 0$, we note that the four-partite thermal entanglement shows robustness similar to the bipartite case. That is, the anisotropy and magnetic field may together be used to produce multipartite entanglement for any finitely large T by adjusting the external magnetic field beyond some finitely large critical strength. Furthermore, we show that this can be understood in the same way as in the two-qubit case. Namely, in terms of the large η behaviors of both the MKB concurrence $\mathcal{C}^{(4)}[|\Phi^{g_0}\rangle\langle\Phi^{g_0}|]$ (equation (12)) and the statistical weight w_{g_0} (equation (22)). When $w_{g_0} \approx 1$, we may conclude that the entanglement associated with χ_4 is of the genuine four-partite kind [15, 16]. Since this is only one kind of the entanglement described by $\mathcal{C}_4$, $\mathcal{C}_4$ should behave similarly to $\mathcal{C}^{(4)}$. In contrast, for $\gamma = 0$ and large η, the ground state is the product state $|1111\rangle$. This important distinction between the four-qubit XX model and the XY model explains why no such robustness of entanglement could be observed for the case of four-qubit isotropic XX chain.

4.3. $N = 6$

We may apply the above analysis employing $\mathcal{C}^{(n)}$ to study XY chains with any even number n of qubits. Firstly, we determine if the ground state $|\Phi^g\rangle$ remains genuinely multipartite entangled at large η. Secondly, we determine if the ground state statistical weight w_g can be made very close to 1 at large η. Through equation (8), these two properties together guarantee that for any finite temperature, non-zero genuine multipartite entanglement can always be obtained by applying a large enough magnetic field. We show that the six-qubit XY

chain has both properties in this subsection. If η is large enough, the minimum eigenvalue of H_6 is given by $E_g = -\lambda J$, where

$$\lambda \equiv \sqrt{3(2+2\gamma^2+\eta^2)+2\kappa+2\sqrt{2}\sqrt{(4\gamma^2+\eta^2)((3+\gamma^2+\eta^2)+\kappa)}}, \tag{24}$$

with $\kappa \equiv \sqrt{\gamma^4+(-3+\eta^2)^2+2\gamma^2(3+\eta^2)}$. The corresponding eigenstate is given by

$$\begin{aligned}|\Phi^g\rangle = N(&|000000\rangle\Theta_1 + |000011\rangle\Theta_2 + |000101\rangle\Theta_3 + |000110\rangle\Theta_2 + |001001\rangle\Theta_4\\ &+ |001010\rangle\Theta_3 + |001100\rangle\Theta_2 + |001111\rangle\Theta_5 + |010001\rangle\Theta_3 + |010010\rangle\Theta_4\\ &+ |010100\rangle\Theta_3 + |010111\rangle\Theta_6 + |011000\rangle\Theta_2 + |011011\rangle\Theta_7 + |011101\rangle\Theta_6\\ &+ |011110\rangle\Theta_5 + |100001\rangle\Theta_2 + |100010\rangle\Theta_3 + |100100\rangle\Theta_4 + |100111\rangle\Theta_5\\ &+ |101000\rangle\Theta_3 + |101011\rangle\Theta_6 + |101101\rangle\Theta_7 + |101110\rangle\Theta_6 + |110000\rangle\Theta_2\\ &+ |110011\rangle\Theta_5 + |110101\rangle\Theta_6 + |110110\rangle\Theta_7 + |111001\rangle\Theta_5 + |111010\rangle\Theta_6\\ &+ |111100\rangle\Theta_5 + |111111\rangle\Theta_8),\end{aligned} \tag{25}$$

where

$$\begin{aligned}
N &= 1/\sqrt{\Theta_1^2+\Theta_8^2+3\left(2\Theta_2^2+2\Theta_3^2+\Theta_4^2+2\Theta_5^2+2\Theta_6^2+\Theta_7^2\right)},\\
\Theta_1 &\equiv \frac{\{24J^2-(\lambda-3J\eta)(\lambda-J\eta)\}\Theta_8+2J^2(\lambda-J\eta)\zeta+8J^2\tau}{2J\gamma(\lambda+3J\eta)},\\
\Theta_2 &\equiv -\frac{(\lambda+3J\eta)\Theta_1}{6J\gamma},\\
\Theta_3 &\equiv -\frac{(\lambda+3J\eta)\Theta_8+(\lambda-J\eta)\tau+2J^2\zeta}{6J\gamma^2},\\
\Theta_4 &\equiv \frac{3(2-\gamma^2)\Theta_8+(\lambda-J\eta)\zeta+2\tau}{3\gamma^2},\\
\Theta_5 &\equiv -\frac{(\lambda-3J\eta)\Theta_8}{6J\gamma},\\
\Theta_6 &\equiv \frac{3\Theta_8+\tau}{3\gamma},\\
\Theta_7 &\equiv -\frac{(\lambda-3J\eta)\Theta_8-2J^2\zeta}{6J\gamma},\\
\Theta_8 &\equiv \lambda^4-2J^2\lambda^2(2+2\gamma^2+\eta^2)+J^4\{\eta^2(\eta^2-12)+4\gamma^2(\eta^2+4)\},
\end{aligned}$$

with $\tau \equiv -8J\lambda\eta\{\lambda^2-J^2(6-2\gamma^2+\eta^2)\}$ and $\zeta \equiv 4\lambda\{\lambda^2(\gamma^2-1)-J^2(4\gamma^4-9\eta^2-4\gamma^2+\gamma^2\eta^2)\}$. The MKB concurrence $\mathcal{C}^{(6)}$ is calculated for this state, giving

$$\mathcal{C}^{(6)}[|\Phi^g\rangle\langle\Phi^g|] = 2N^2|\Theta_1\Theta_8+6\Theta_2\Theta_5+6\Theta_3\Theta_6+3\Theta_4\Theta_7|. \tag{26}$$

In the limit of large η

$$\mathcal{C}^{(6)}[|\Phi^g\rangle\langle\Phi^g|] \approx 2\gamma^3\eta^{-3}, \tag{27}$$

which is non-zero if $\gamma \neq 0$. It can again be shown that $|\Phi^g\rangle$ is a genuine six-partite entangled state [15, 16]. In addition, the statistical weight w_g of the ground state $|\Phi^g\rangle$ can be verified numerically to be close to unity when an appropriately large magnetic field is applied. The ground state $|\Phi^g\rangle$ therefore possesses the two key properties, which can be found in both the $N = 2$ and $N = 4$ cases. It may therefore be deduced, via equation (8), that the system could have non-zero genuine six-partite entanglement at any finite temperature, provided an appropriate external magnetic field is applied.

5. Remarks and conclusion

We have investigated in detail the origin of the robustness of multipartite thermal entanglement in two-, four- and six-qubit XY models. In addition, our results complement those in [17], which employed bipartite entanglement measure[6]. Two important properties common to the ground states $|\Phi^g\rangle$ of these models, which give rise to the characteristic robustness are identified. Firstly, the statistical weight w_g corresponding to $|\Phi^g\rangle$ can be made very close to unity by applying a large enough magnetic field. This allows one to conclude via equation (8) that the MKB concurrence of the thermal state χ can be well approximated by that of $|\Phi^g\rangle$. Secondly, and more importantly, $|\Phi^g\rangle$ remains (genuinely) multipartite entangled under such a magentic field. This guarantees that there is non-zero (genuine) multipartite entanglement for any finite temperature when a sufficiently large magnetic field is applied. These properties allow one to extend our study to XY chains with any even number of qubits. XY models with odd number of qubits are not considered here since $\mathcal{C}^{(n)} = 0$ for odd n. However, one could similarly study the robustness of other kinds of entanglement by employing the other MKB concurrences. We conjecture that robust multipartite thermal entanglement can be found in XY chains with any number of particles.

The applicability of our analysis is not only restricted to the XY chains. If the ground state of a physical system can be made to dominate the thermal state at any finite temperature by adjusting some parameters of the system Hamiltonian, then the MKB concurrence of the thermal state can be well approximated by that of its ground state (equation (8)). So, in addition, if the ground state remains multipartite entangled under these conditions, then similar robustness of multipartite entanglement can be expected to be observable. Therefore, our analysis can be used to identify possible candidates (like those considered in [18, 19]) for realization of quantum computation at finite temperatures. Admittedly, the quantity of thermal entanglement may be very small. However, we must point out that for mixed-state quantum computation, a large amount of entanglement does not seem to be necessary to obtain a speedup with respect to classical computers [20], in contrast to pure-state quantum computation.

The MKB concurrences can be directly measured for multipartite pure states, if two copies of the states are available [11, 21]. Since we are interested in the region where the ground state $|\Phi^g\rangle$ is the only state that contributes significantly to the thermal state χ, the state χ is 'almost pure' and therefore it is possible to measure the MKB concurrences of χ with a high degree of success. The results obtained here and the above conjecture may thus, in principle, be experimentally verified.

In conclusion, we have spelt out a rather general method of identifying physical systems where robust multipartite entanglement at finite temperatures may be found. Using the XY model as an example, we have shown how the necessary analysis could be reduced to a study of the system ground states.

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[6] In general, it is not sufficient to only consider bipartite entanglement. For instance, the Greenberger–Horne–Zeilinger (GHZ) state [23] is a state with genuine multipartite entanglement but yields zero entanglement between one particle and any other particle. On the other hand, the W state [14] is one where every particle is entangled with every other particle, but it has no genuine multipartite entanglement [8, 15].

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